Joint use of the Weniger transformation and hyperasymptotics for accurate asymptotic evaluations of a class of saddle-point integrals

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(Received 19 June 2008; published 8 August 2008)

When the Weniger transformation is systematically used in resumming the asymptotic series arising from the application of the steepest descent method can display dramatic numerical instabilities that prevent it from improving the accuracy achievable via superasymptotics. In the present paper an explanation of such pathologies through the concept of resurgence, introduced by Berry and Howls several years ago within the context of hyperasymptotics, is proposed. In particular, the way the topology of the whole set of the saddles influences the resummation capabilities of the Weniger transformation is here investigated for the integrals defining the lowest-order cuspoid diffraction catastrophes. Eventually, a powerful and easily implementable resummation scheme, based on a joint use of the Weniger transformation and hyperasymptotics, is proposed for taking care of the above pathologies. Such a joint action seems to encompass the main virtues of both approaches, and the preliminary numerical results obtained from its application show that relative errors several orders of magnitude smaller than those achievable via superasymptotics can be achieved with modest implementation efforts.

DOI: 10.1103/PhysRevE.78.026703 PACS number(s): 02.60.−x, 02.30.Lt, 02.30.Mv, 02.70.−c

I. INTRODUCTION

Asymptotics finds a natural field of applications at the borderlands between physical theories, such as for instance classical and quantum mechanics, physical and geometrical optics, statistical mechanics and thermodynamics, where finding the solution of several problems requires one to evaluate integrals of the form

$$\mathcal{I}(k) = \int_{\mathcal{C}} g(z)\exp[-kf(z)]dz, \qquad (1)$$

where $\mathcal{C}$ is a suitable integration path in the complex z plane and $g(z)$ and $f(z)$ are functions which, for simplicity, will be assumed to be nonsingular. Moreover, k is customarily intended as a "large" parameter, similar to the wave number in physical optics, or the inverse of the Planck's constant in quantum mechanics, or even the Reynold's number in fluid mechanics [1]. While it is a well-known fact that the leading term of the asymptotic expansion, with respect to $1/k$, of integrals such as those in Eq. (1) can be obtained by simply summing all the leading contributions coming from the different saddle points of $f(z)$ belonging to the integration path [2], likely it is less known that all higher-order terms can be systematically evaluated and arranged in the form of power series, following a procedure introduced by Dingle [3]. The main feature of Dingle series is represented by their divergent character, which is much more evident whenever the evaluation of $\mathcal{I}(k)$ is attempted for "nonlarge" values of $|k|$, i.e., beyond asymptotics.

In the present paper we are interested in studying accurate numerical evaluations of integrals of the type in Eq. (1) by means of resummation of the divergent series associated with the contributions coming from each saddle belonging to $\mathcal{C}$. Among the numerical techniques developed, since Euler's time, for the resummation of divergent series [4], our interest is focused on the so-called δ or Weniger transformation (WT for short) proposed, at the end of the 1980s, by Weniger [5] to convert the sequences of the partial sums of factorial divergent series into new sequences, quickly converging toward the limit of the starting series. Since its introduction, the WT has successfully been applied for solving several problems in quantum mechanics [6–9] and, more recently, its use has also been proposed within the optical realm [10–13]. Concerning the use of the WT for evaluating integrals in Eq. (1), the idea is quite simple: since the asymptotic series associated to each contributive saddle display a factorial divergent character [3], and due to the fact that the WT has been designed especially for dealing with series endowed with such type of divergence, we should expect the value of $\mathcal{I}(k)$ to be quickly and accurately retrieved with a modest computational effort. Unfortunately, while in several situations the above-described scenario corresponds to the truth, there are some pathological, but nevertheless important, cases in which the WT revealed to be unable to accomplish the desired resummation process. In a sense, such failures reflect the extreme "specialization" of the WT which, for the resummation of factorial divergent series to be successfully achieved, requires an alternating character of the sequence of its single terms [14]. A simple but important example of the above pathology is given by the asymptotic evaluation of the Airy function in the neighborhood of a Stokes line, for which it is easy to show that the corresponding asymptotic factorial series becomes *nonalternating*. As a consequence, it turns out to be no longer resummable via the WT [14,15]. Moreover, different pathological situations, not attributable to a pure nonalternating divergent character of the asymptotic series, in which the WT fails in achieving the resummation can still occur, an example of these being furnished by the evaluation of the Pearcey function for certain choices of its arguments.

The aim of the present paper is twofold. First, a way of interpreting the possible failures of the WT is proposed in terms of the so-called *resurgence* property discovered, in the context of asymptotics, by Berry and Howls [16,17]. With this approach, the divergent character of the asymptotic series arising from the steepest-descent treatment of Eq. (1) was ascribed to the presence of other saddles, *not belonging* to the integration path $\mathcal{C}$ [17]. As a matter of fact, the asymptotic, i.e., for large indexes, behavior of the expanding coefficients turns out to be strongly influenced by the topology of the *whole* set of saddles of $f(z)$. By using this strategy some pathological situations in which the WT is expected to fail can then easily be identified. As a second, more operative, task, the present paper is pursuing the hope that the resummation capabilities of the WT, together with its remarkably implementation ease, can still be employed even in the presence of pathological saddle topologies. The idea consists in trying to operate a sort of "regularization" on the single terms of the original pathological diverging series in order for them to acquire the required alternating character. We propose as a possible simple way to achieve such a regularization the first stage of the so-called *hyperasymptotics* (H for short) [16,17]. In particular, by carrying out a series of preliminary numerical experiments concerning the asymptotic evaluation, via Eq. (1), of the Airy and the Pearcey functions, it will be found that such a joint use of H and of the WT (H-WT henceforth) allows relative errors comparable with those achievable by the WT for nonpathological cases (corresponding to relative errors several orders of magnitude smaller than those achievable via superasymptotics) to be obtained with modest computational efforts and implementation ease of the same level as in the WT alone.

To keep the paper reasonably self-consistent, in the next section a brief review of the Dingle method within the steepest descent treatment of integrals in Eq. (1) is provided, together with a basic description of the WT.

II. REVIEW OF THE DINGLE METHOD AND OF THE WT

A. Steepest descent and the Dingle method

For simplicity, we suppose the two functions $f(z)$ and $g(z)$ to be analytic in the complex plane. Furthermore, all saddle points, say $\{z_n\}$, which are defined through the equation

$$\left.\frac{df(z)}{dz}\right|_{z=z_n}=0, \tag{2}$$

are supposed to be simple, i.e., such that $f''(z_n)\neq 0$ for any n. The set of the saddle points of $f(z)$ will be denoted $\mathcal{S}$, and the integration path $\mathcal{C}$ will be thought of, thanks to the nonsingularity of $g(z)$ and $f(z)$, as the union of a finite number of steepest descent arcs, each of them, say $\mathcal{C}_n$, passing through the contributive saddle point z_n. Accordingly, the quantity $\mathcal{I}(k)$ can generally be written as

$$\mathcal{I}(k)=\int_{\mathcal{C}} g(z)\exp[-kf(z)]dz=\sum_{n\in\mathcal{S}'}\mathcal{I}^{(n)}(k), \tag{3}$$

where $\mathcal{S}'$ denotes the subset of $\mathcal{S}$ containing all the contributive saddles and

$$\mathcal{I}^{(n)}(k)=\int_{\mathcal{C}_n} g(z)\exp[-kf(z)]dz. \tag{4}$$

The last integral can be written as [17]

$$\mathcal{I}_n(k)=k^{-1/2}\exp(-kf_n)T^{(n)}(k), \tag{5}$$

where $f_n=f(z_n)$, and where $T^{(n)}(k)$ can *formally* be written through the following asymptotic series expansion:

$$T^{(n)}(k)=\sum_{r=0}^{\infty}k^{-r}T_r^{(n)}, \tag{6}$$

the expanding coefficients $T_r^{(n)}$ being expressed via the integral representation [17]

$$T_r^{(n)}=\frac{(r-1/2)!}{2\pi i}\oint_n \frac{g(z)}{[f(z)-f_n]^{r+1/2}}dz. \tag{7}$$

Note that the subscript n denotes a small positive loop around the saddle z_n, and, in the neighborhood of z_n, $f(z)$ can always be expanded as

$$f(z)-f_n=(z-z_n)^2U_n(z), \tag{8}$$

where $U_n(z)$ is a function which turns out to be regular at z_n. By substituting from Eq. (8) into Eq. (7), and by using Cauchy theorem, it is found at once that

$$T_r^{(n)}=\frac{(r-1/2)!}{(2r)!}\left\{\frac{d^{2r}}{du^{2r}}\frac{g(u+z_n)}{[U_n(u+z_n)]^{r+1/2}}\right\}_{u=0}. \tag{9}$$

The series expansion in Eq. (6) is asymptotic with respect to $1/k$. This implies that, for large values of $|k|$, estimates of $\mathcal{I}^{(n)}(k)$ with sufficient accuracy can be achieved by taking into account only a few terms of the series. However, things are much more troublesome and delicate when such series are attempted to be used beyond the asymptotic regime, as in our case. In this case, in fact, the factor $(r-1/2)!$ in Eq. (9) dominates for large values of r and, as a consequence, the sequence of the partial sums of the series diverges following a factorial law.

B. The Weniger transformation

An important feature of several classes of factorial divergent series is that they can be efficiently decoded by using suitable nonlinear transformation schemes. Such transformations, once acting on the sequence of the partial sums, are able to convert it into new sequences which quickly converge to the so-called *antilimit* of the starting diverging series. As hinted in the introduction, for factorial divergence the δ or Weniger transformation (WT for short) represents one of the most efficient resummation transformations [5]. Generally speaking the WT, when applied to the sequence of the partial sums, say $\{S_n\}$, $S_n=\Sigma_{j=0}^n a_j$ ($n=0,1,\ldots$) of a series, converts it into a new sequence, say $\{\delta_k\}$ ($k=1,2,\ldots$), defined by [5]

$$\delta_k=\frac{\displaystyle\sum_{j=0}^{k}(-1)^j\binom{k}{j}(1+j)_{k-1}\frac{S_j}{a_{j+1}}}{\displaystyle\sum_{j=0}^{k}(-1)^j\binom{k}{j}(1+j)_{k-1}\frac{1}{a_{j+1}}}, \tag{10}$$

where $(\cdot)_k$ denotes the Pochhammer symbol [18]. For a full and rigorous presentation of the theoretical basis of nonlinear transformations, and of the WT in particular, the interested reader is encouraged to consult the available large bibliography and, in particular, Refs. [5,19,20]. As previously said, our aim is to investigate the possibility of using WT even in the presence of pathological situations, leading for instance to nonalternating series, for which the direct use of the WT does not allow the integral in Eq. (1) to be correctly retrieved. In the next section we show two examples of asymptotic evaluation of phase integrals in which the above pathologies are clearly manifested.

III. TWO EXAMPLES OF PATHOLOGICAL SITUATIONS FOR THE WT

A. Preliminaries

In the present section we are going to show a couple of examples concerning the asymptotic evaluation of phase integrals of the type in Eq. (1) for which the direct use of the WT leads to numerical instabilities. The examples we are going to deal with concerns two special functions playing a role of pivotal importance in optics and quantum mechanics, namely the Airy and the Pearcey functions.

B. Airy function

Consider the evaluation of the Airy function defined by

$$\mathrm{Ai}(x) = \frac{1}{2\pi}\int_{\mathcal{C}} \exp\left[i\left(\frac{z^3}{3}+xz\right)\right]dz, \tag{11}$$

which is of the form given in Eq. (1) with $g(z)=1/(2\pi)$, $f(z)=-i(z^3/3+xz)$, and $k=1$. There are two saddles, say $z_\pm$, given by $z_\pm=\pm(-x)^{1/2}$. The path $\mathcal{C}$ consists of the steepest descent arcs which, for $|z|\to\infty$, must necessarily asymptotically approach the directions $\varphi=(2n+1/2)\pi/3$, with $n=0,1,2$. In particular, when $\arg\{x\}<2\pi/3\mathcal{C}=\mathcal{C}_+$, i.e., only one saddle (and precisely z_+) does contribute to the integral, i.e., $\mathrm{Ai}(x)=\mathcal{I}^{(+)}(1)$. On the contrary, when $\arg\{x\}>2\pi/3$ it turns out that $\mathcal{C}=\mathcal{C}_+\cup\mathcal{C}_-$, i.e., even the saddle z_- becomes *fully* contributive, so that $\mathrm{Ai}(x)=\mathcal{I}^{(+)}(1)+\mathcal{I}^{(-)}(1)$. Finally, when $\arg\{x\}=2\pi/3$, i.e., at the Stokes line, the contribution to the total integral coming from z_- must be weighted with a factor $1/2$, so that $\mathrm{Ai}(x)=\mathcal{I}^{(+)}(1)+\mathcal{I}^{(-)}(1)/2$ [21]. As far as the expanding coefficients are concerned, we have that $U_\pm(z)=(z+3z_\pm)/3i$ so that, from Eq. (9), after straightforward algebra it is found that

$$T_r^{(\pm)} = \frac{1}{2\sqrt{\pi}}\frac{i^{r+1/2}}{z_\pm^{3r}}\left(\frac{1}{36}\right)^r \frac{\left(3r-\frac{1}{2}\right)!}{r!\left(r-\frac{1}{2}\right)!}. \tag{12}$$

In Table I the results of a first numerical experiment, concerned with the numerical evaluation of the Airy function for $x=(3/4\times 16)^{2/3}$, are presented. The same experiment was already considered in Ref. [16], whose results will be used later for comparison purposes. It is seen how, in the present case, 14 single terms of the original series are sufficient to retrieve the values of the Airy function up to nearly 20 figures. Although, by quoting Berry, "… *I would like to hear from anybody who needs the Airy function to 20 decimals, but am not expecting an early call*" [22], the results given in

TABLE I. The action of the WT on the sequence of the partial sum of the asymptotic series of the Airy function Ai(x) for Airy function for $x=(3/4\times 16)^{2/3}$. First column: sequence index. Second column: estimate provided by the WT. Third column: relative error. The number of digits used in the numerical calculations was set to 20.

n	Weniger δ_n	Rel. error
2	$6.2033672725872058228\times 10^{-5}$	3×10^{-5}
3	$6.2031902884950443666\times 10^{-5}$	10^{-6}
4	$6.2032021455073140033\times 10^{-5}$	10^{-7}
5	$6.2032014734562527659\times 10^{-5}$	5×10^{-9}
6	$6.2032015096234582762\times 10^{-5}$	3×10^{-10}
7	$6.2032015077493629066\times 10^{-5}$	10^{-11}
8	$6.2032015078412871931\times 10^{-5}$	6×10^{-13}
9	$6.2032015078371437062\times 10^{-5}$	3×10^{-14}
10	$6.2032015078373072958\times 10^{-5}$	8×10^{-16}
11	$6.2032015078373021565\times 10^{-5}$	2×10^{-17}
12	$6.2032015078373022499\times 10^{-5}$	2×10^{-19}
13	$6.2032015078373022516\times 10^{-5}$	10^{-19}
14	$6.2032015078373022514\times 10^{-5}$	0

Table I provide a realistic idea about the resummation capabilities of the WT. Consider now, as a second experiment, the evaluation of the Airy function at the Stokes line, i.e., for $|x|=(3/4\times 16)^{2/3}$ and $\arg\{x\}=2\pi/3$. From what is said above, both the contributions $\mathcal{I}^{(\pm)}(1)$ have to be evaluated. As far as $\mathcal{I}^{(-)}(1)$ is concerned, it turns out that 13 single terms are sufficient to obtain an estimate accurate up to 20 figures, but the same is not true for the contribution $\mathcal{I}^{(+)}(1)$. The results are shown in Table II, where the sole relative error is reported as a function of the WT order n. It is clearly

TABLE II. Instability in WT across the Stokes line for the Airy function. The same as in Table I, but for $x=(3/4\times 16)^{2/3}\exp(i2\pi/3^{-})$. Note that only relative error is shown. The relative error obtained with the superasymptotic approximation is about 6×10^{-8}.

n	Rel. error
2	5×10^{-5}
3	7×10^{-6}
4	10^{-6}
5	3×10^{-7}
6	10^{-7}
7	3×10^{-8}
8	6×10^{-9}
9	7×10^{-8}
10	2×10^{-7}
11	2×10^{-8}
12	6×10^{-8}
13	10^{-7}
14	3×10^{-9}
15	2×10^{-7}

seen how in this case the WT is unable to improve the accuracy of the estimate of the Airy function obtained by using standard superasymptotics [22,23], i.e., by truncating the starting diverging series at its least term, which provides a relative error of 6×10^{-8}.

C. Pearcey function

The Pearcey function $P(x,y)$ is defined through the equation

$$P(x,y)=\int_{\mathcal{C}}\exp\left[i\left(\frac{z^4}{4}+x\frac{z^2}{2}+yz\right)\right]dz, \qquad (13)$$

where x and y are complex numbers [24]. By comparing Eqs. (13) and (1), the Pearcey function corresponds to the choice $k=1$, $g(z)=1$, and $f(z)=-i(z^4/4+xz^2/2+yz)$. As far as the steepest descent path $\mathcal{C}$ is concerned, since $f(z)\simeq -iz^4$ for large $|z|$, it must asymptotically approach the directions $\varphi=(2n+1/2)\pi/4$, with $n=0,\ldots,3$, while the position of the three saddles, say z_n $(n=1,\ldots,3)$, can be analytically obtained from Eq. (2) via standard algebraic formulas. Moreover, the expanding coefficients $T_r^{(n)}$ corresponding to the saddle z_n are given by the following closed-form expression [17]:

$$T_r^{(n)}=\frac{\sqrt{2}\,i^{r+1/2}\left(r-\frac{1}{2}\right)!}{z_n^{4r+1}\left(3+\frac{x}{z_n^2}\right)^{2r+1/2}}C_{2r}^{r+1/2}\left(\sqrt{\frac{2}{3+x/z_n^2}}\right), \qquad (14)$$

where $C_r^m(\cdot)$ denotes the Gegenbauer polynomial [18]. Of course, also in the case of the Pearcey function the Stokes phenomenon occurs. For instance, in the case of *real* values of the parameters (x,y), Wright found that a Stokes set is defined by the surface of equation $27y^2-(5+3\sqrt{3})x^3=0$ [25]. From what is said above, we expect that the WT will fail in evaluating the Pearcey function there. Nevertheless, in the present section we are more interested to show sources of instabilities for the WT which are not directly connected to Stokes phenomena, which will be considered later. In doing so, we consider first the evaluation of the Pearcey function at the complex pair $(x,y)=(7,1+i)$. The same numerical experiment was already considered in Ref. [17], and we shall use later the corresponding results for comparison purposes. For the above choice of the pair (x,y), the corresponding saddle topology leads to a steepest descent integration path involving only one of the three saddles, so that only one asymptotic series has to be evaluated. Of the three saddles, namely $z_1\simeq 0.0776212\ldots-i2.57482\ldots$, $z_2\simeq -0.143675\ldots -i0.14201\ldots$, and $z_3\simeq 0.066054\ldots+i2.71683\ldots$, only z_2 contributes to the asymptotic evaluation of the Pearcey function, i.e., $P(x,y)=\mathcal{I}^{(2)}(1)$. In Table III, the relative error, obtained with respect to the 15-figure "exact" value provided in Ref. [17], is reported as a function of the WT order n.

It is seen how, in the present case, the WT is able to achieve a relative error of about 10^{-12} by using 13 terms, whereas 24 terms are sufficient to fully retrieve the "exact" 15-digit value. A radically different situation is displayed for the pair $(x,y)=(7,1/\sqrt{2})$, which has also recently been examined in Ref. [26] by using a hyperasymptotic approach based on Hadamard expansions. As for the previous case, of the three saddles $z_1\simeq 0.0504343\ldots-i2.64719\ldots$, $z_2\simeq -0.100869\ldots$, and $z_3=z_1^*\simeq 0.0504343\ldots+i2.64719\ldots$, only z_2 contributes to the asymptotic evaluation of the Pearcey function, so that also in this case $P(x,y)=\mathcal{I}^{(2)}(1)$. However, now the behavior of the relative error versus n, which has been evaluated with respect to the 20-digit "exact" value provided in Ref. [26] reveals the inability of the WT to go beyond the error level, about 6×10^{-8}, obtained through the superasymptotic least-term truncation, as reported in Table IV.

TABLE III. The same as in Table I, but for the numerical evaluation of the Pearcey function at the complex pair $(x,y)=(7,1+i)$. The relative error is evaluated with respect to the 15-digits "exact" value provided in Ref. [17]. The number of digits has been fixed to 15.

n	Weniger δ_n	Rel. error
2	$0.788896466486375+0.751997407638978i$	10^{-4}
3	$0.788914710309079+0.752114814531054i$	10^{-5}
4	$0.788924392759829+0.752103988027836i$	10^{-6}
5	$0.788922823746154+0.752103897014747i$	6×10^{-8}
6	$0.788922864549319+0.752103946845074i$	3×10^{-8}
7	$0.788922838583707+0.752103952575956i$	7×10^{-9}
8	$0.788922837830415+0.752103958064321i$	2×10^{-9}
9	$0.788922837548435+0.752103959286292i$	4×10^{-10}
10	$0.788922837601425+0.752103959634780i$	10^{-10}
11	$0.788922837602631+0.752103959721517i$	4×10^{-11}
12	$0.788922837599937+0.752103959746942i$	10^{-11}
13	$0.788922837597681+0.752103959755574i$	3×10^{-12}
14	$0.788922837597076+0.752103959758409i$	8×10^{-13}
15	$0.788922837597028+0.752103959759183i$	8×10^{-14}
16	$0.788922837597044+0.752103959759296i$	8×10^{-14}
17	$0.788922837597029+0.752103959759270i$	6×10^{-14}
18	$0.788922837597002+0.752103959759244i$	3×10^{-14}
19	$0.788922837596983+0.752103959759236i$	10^{-14}
20	$0.788922837596973+0.752103959759237i$	7×10^{-15}
21	$0.788922837596970+0.752103959759239i$	3×10^{-15}
22	$0.788922837596969+0.752103959759241i$	10^{-15}
23	$0.788922837596969+0.752103959759242i$	10^{-15}
24	$0.788922837596969+0.752103959759243i$	0

It should be noted how a rather similar pathological situation is displayed (not shown for brevity in the present section) when the Pearcey function is evaluated via the WT at the pair $(x,y)=(-4,12/\sqrt{2})$, in which case the set $\mathcal{S}'$ consists of *two* contributive saddles [26]. Also this case will be considered later for comparison purposes.

D. Discussion

The above-described numerical experiments give evidence about the inability of the WT in doing the job in certain pathological situations. From a mere computational

TABLE IV. The same as in Table III, but for the numerical evaluation of the Pearcey function at the pair $(x,y)=(7,1/\sqrt{2})$. The relative error is evaluated with respect to the 20-digits "exact" value provided in Ref. [26].

n	Rel. error
2	8×10^{-5}
3	8×10^{-6}
4	6×10^{-7}
5	2×10^{-8}
6	10^{-8}
7	9×10^{-9}
8	8×10^{-9}
9	10^{-8}
10	6×10^{-8}
11	6×10^{-8}
12	6×10^{-8}
13	6×10^{-8}
14	6×10^{-8}
15	6×10^{-8}
…	
20	7×10^{-8}
21	7×10^{-8}
22	7×10^{-8}
23	6×10^{-8}

viewpoint, they could be ascribed to the particular mathematical structure of Eq. (10), which, as explained in Ref. [14], was explicitly conceived for resumming *alternating* factorial divergent series. The price to be paid for such a "specialization" is that it tends to become numerically highly unstable in presence of deviations from this divergence law [14]. This was particularly evident for the evaluation of the Airy function on the Stokes set, when the asymptotic series defining $\mathcal{I}^{(+)}(1)$ turned out to be no longer alternating, as can easily be checked by letting $x=\rho\exp(i2\pi/3)$ into Eq. (12), which led to $T_r^{(+)}>0$ for *any* r. The explanation of the WT failure in the Pearcey function evaluation described in Sec. III C requires a deeper understanding of the mechanisms which generate the divergence of the asymptotic series. As we shall see in the next section, a simple interpretation of the above-described WT pathologies can be given in terms of the so-called resurgence property discovered, in the context of the asymptotics, by Berry and Howls at the beginning of 1990s [16,17]. Following this approach, in the next section it will be shown how the above failures of the WT for the asymptotic series resummation can directly be ascribed to particular saddle topologies associated to the function $f(z)$, which lead to asymptotic laws for the expanding coefficients not suitable for the WT to successfully operate the resummation.

IV. USING RESURGENCE TO INTERPRET THE FAILURE OF THE WT

For the sake of clarity, it is worth briefly reviewing the concept of resurgence. We shall strictly follow the notations of Ref. [17]. Loosely speaking, Berry and Howls discovered that the divergent character of the asymptotic series in Eq. (6) can be interpreted as due to the presence of other saddles, say z_m, with $n\neq m$, which do not belong to the steepest descent path $\mathcal{C}_n$. Such saddles, called "adjacent" to z_n, must be identified via a topological rule. The starting point is again Eq. (6), which is now rewritten in the alternative form [27]

$$T^{(n)}(k)=\sum_{r=0}^{N-1}k^{-r}T_r^{(n)}+R^{(n)}(k,N), \tag{15}$$

where N represents a truncation index and $R^{(n)}(k,N)$ denotes the corresponding remainder. The analysis of Ref. [17] started from the following integral representation of the remainder:

$$R^{(n)}(k,N)=\frac{1}{2\pi i k^N}\int_0^\infty du\,\exp(-u)u^{N-1/2}\times\oint_{\Gamma_n}dz\frac{g(z)}{[f(z)-f_n]^{N+1/2}\left\{1-\dfrac{u}{k[f(z)-f_n]}\right\}}, \tag{16}$$

where Γ_n denotes a "sausage" positive infinite loop encircling the steepest descent path $\mathcal{C}_n$ [17,28]. The next step is to let Γ_n expanding in such a way for it to intercept those saddles which are *adjacent* to z_n. We shall denote by $\mathcal{A}_n$ the set containing the indexes pertinent to all saddles adjacent to z_n. In the cases we have considered in Sec. III, the corresponding sets of adjacent saddles are immediately identified. For instance, in the case of the Airy function, where only two saddles, $z_\pm$, are involved, it turns out that $\mathcal{A}_\pm=\{\mp\}$. On the contrary, in the case of the Pearcey function the set $\mathcal{A}_n$ depends on the choice of the pair (x,y). Thanks to the analyticity of f and g, the expanded loop Γ_n can be deformed into the union of arcs at infinity and arcs passing through *all* adjacent saddles z_m with $m\in\mathcal{A}_n$. Nontrivial algebra eventually gives for the remainder $R^{(n)}(k,N)$ [17]

$$R^{(n)}(k,N)=\frac{1}{2\pi i}\sum_{m\in\mathcal{A}_n}\frac{(-1)^{\gamma_{nm}}}{(kF_{nm})^N}\times\int_0^\infty dv\frac{v^{N-1}\exp(-v)}{1-\dfrac{v}{kF_{nm}}}T^{(m)}\left(\frac{v}{F_{nm}}\right), \tag{17}$$

where the quantities F_{nm}, called *singulants*, are given by

$$F_{nm}=f_m-f_n, \tag{18}$$

and the binary quantities $\gamma_{nm}\in\{0,1\}$ are obtained through a topological rule [17]. Resurgence followed from Eq. (17) in the form of a relationship between the expanding coefficients at the saddle z_n, $T_r^{(n)}$, and those at all its adjacent saddles, $T_r^{(m)}$ ($m\in\mathcal{A}_n$). More precisely, it was found that [17]

$$T_r^{(n)} = \frac{1}{2\pi i} \sum_{m \in \mathcal{A}_n} (-1)^{\gamma_{nm}} \sum_{l=0}^{\infty} \frac{(r-l-1)!}{F_{nm}^{r-l}} T_l^{(m)}, \tag{19}$$

which, although in principle being only formal (because the factorials are infinite as $l > r-1$), constitutes a possible way for interpreting the failures of the WT above illustrated. In fact, consider the asymptotic expression, for large r, of $T_r^{(n)}$ which, from Eq. (19), turns out to be

$$T_r^{(n)} \approx \frac{(r-1)!}{2\pi i} \sum_{m \in \mathcal{A}_n} \frac{(-1)^{\gamma_{nm}} T_0^{(m)}}{F_{nm}^r}. \tag{20}$$

If, among the adjacent saddles z_m, there is just one, say $z_{\bar{m}}$, called *dominant*, such that $F_{n\bar{m}}$ takes the *minimum* modulus, then Eq. (20) can be approximated by

$$T_r^{(n)} \approx \frac{(-1)^{\gamma_{n\bar{m}}} T_0^{(\bar{m})}}{2\pi i} \frac{(r-1)!}{F_{n\bar{m}}^r}, \tag{21}$$

which corresponds to a pure asymptotic factorial divergence law for the late terms of the series in Eq. (6). When the Airy function has been evaluated in Sec. III B across the Stokes line, we found that the contribution $\mathcal{I}^{(+)}(1)$ was not properly retrieved. This happened because $\mathcal{A}_+ = \{-\}$ and the singulant F_{+-}, which is given by

$$F_{+-} = f_- - f_+ = -\frac{4}{3} x^{3/2}, \tag{22}$$

for $x = |x| \exp(i 2\pi/3)$ turns out to be real and *positive*. Accordingly, from Eq. (21) the asymptotic expansion of $\mathcal{I}^{(+)}(1)$ is a factorial divergent *nonalternating* character, as we already found through the elementary analysis of Sec. III B.

Concerning the Pearcey function, we refer to the results reported in Tables III and IV, respectively. In both cases $P(x,y) = \mathcal{I}^{(2)}(1)$ and $\mathcal{A}_2 = \{1,3\}$. However, for the case of Table III *there is* a dominant adjacent saddle to z_2, and precisely z_1, being $|F_{21}| \simeq 9.90429\ldots$ and $|F_{23}| \simeq 15.1688\ldots$. Accordingly, Eq. (21) applies and, due to the fact that F_{21} assume a *complex* value, the resulting series asymptotically diverges with an alternating factorial character, thus allowing the WT to quickly converge. The situation is different for the experiment of Table IV, where $z_3 = z_1^*$, so that, since $F_{23} = -F_{21}^*$, *both* adjacent saddles do contribute to the asymptotic expression of the late term $T_r^{(2)}$. In particular, from Eq. (14) it can be possible to show that $T_0^{(3)} = i[T_0^{(1)}]^*$, so that Eq. (20) eventually leads to

$$T_r^{(2)} \approx (-) \frac{(r-1)!}{2\pi i} \left\{ \frac{T_0^{(1)}}{F_{21}^r} + i \left[\frac{T_0^{(1)}}{F_{21}^r} \right]^* \right\}, \tag{23}$$

where the fact that $\gamma_{21} = 0$ and $\gamma_{23} = 1$ has been taken into account [17]. It is interesting to compare the approximate behavior, as a function of r, of the modulus of $T_r^{(2)}$ given by the asymptotic law in Eq. (23), with that obtained from the exact expression in Eq. (14), which are reported, as a solid curve and dots, respectively, in Fig. 1. Accordingly, we conclude that the expanding coefficients $T_r^{(2)}$ do not display the asymptotic alternating factorial divergence of Eq. (21), so that WT is expected to fail.

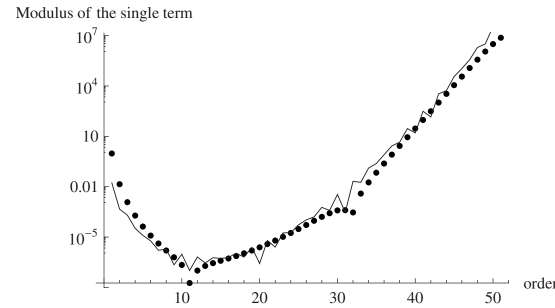

FIG. 1. Behaviors, as functions of the order r, of the modulus of $T_r^{(2)}$ given by Eq. (14) (dots) and that given by the asymptotic law in Eq. (23) (solid curve) for the pair $(x=7, y=1/\sqrt{2})$.

V. THE *H*-WT RESUMMATION PROCEDURE

In the previous section the resurgence concept has successfully been used to give a possible "topological" interpretation of the failures of the WT described in Sec. III. Resurgence, however, has also operative implications, it being the basis of the so-called *hyperasymptotics* (H for short) [16,17]. Differently from the WT, H was not conceived as a mere resummation tool, but rather as an elegant interpretation scheme of the above described divergences based on a, nearly aesthetic, iterated optimal truncation, or superasymptotic, principle. Very roughly speaking, the action of H can be summarized as follows: starting from the sequence $\{T_r^{(n)}\}$, a first optimal truncation is operated, leading to the superasymptotic estimate. With reference to Eq. (15), this means to truncate the series at its least term. As far as the (divergent) tail, represented by the remainder $R^{(n)}(k,N)$, is concerned, by using the resurgence property in Eq. (17), after long and nontrivial analysis, Berry and Howls found that it can formally be rewritten as a sum of asymptotic series whose single terms involved the expanding coefficients $T_r^{(m)}$ pertinent to *all* adjacent saddles, i.e., with $m \in \mathcal{A}_n$. More precisely, substitution from Eq. (6) into Eq. (17) led, after nontrivial algebra, to the following expression for the remainder [17]:

$$R^{(n)}(k,N) = \frac{(-1)^N}{2\pi i} \sum_{m \in \mathcal{A}_n} (-1)^{\gamma_{nm}} \times \sum_{r=0}^{\infty} \frac{(-1)^r}{k^r} T_r^{(m)} K_{N-r}^{(1)}(-kF_{nm}), \tag{24}$$

where the function $K_n^{(1)}(\beta)$, called *terminant* of order 1 [16,17], is defined through the integral

$$K_n^{(1)}(\beta) = \frac{1}{\beta^n} \int_0^{\infty} dv \frac{v^{n-1} \exp(-v)}{1 + \dfrac{v}{\beta}}, \tag{25}$$

where, in order for it to converge, $n > 0$. Note that the integral in Eq. (25) can be expressed through the closed-form expression

$$K_n^{(1)}(\beta)=\exp(\beta)\left[\frac{E_n(\beta)}{\beta^{n-1}}(n-1)!+(-1)^{n-1}i\pi\epsilon\right], \quad (26)$$

where $E_n(\cdot)$ denotes the exponential integral function, while ϵ equals 1 if $\beta<0$ and zero otherwise. From a mere mathematical viewpoint, the presence of the term containing ϵ has to be ascribed to the evaluation of the integral in Eq. (25), for $\beta<0$, in the Cauchy principal-value sense. As we shall see in the next section, this is a crucial point for allowing the WT to be still used across the Stokes sets. Equation (24) represents the first hyperasymptotic stage, at which the divergence of the series in Eq. (6) is led back to the presence of adjacent saddles [17]. Of course, the asymptotic series in Eq. (24) are only formal, since for $r>N$ the terminant $K_{N-r}^{(1)}$ diverges. The main idea of H consists in superasymptotically truncating each of the asymptotic series in Eq. (24) at optimal truncation indexes $N_m<N$, with $m\in\mathcal{A}_n$. In this way a first hyperasymptotic correction is obtained which, once summed to the above described superasymptotic estimate, provides a first hyperasymptotic estimate for $\mathcal{I}^{(n)}$. Of course, in doing so, for each adjacent saddle z_m a new divergent remainder will be generated. By iterating the above optimal truncation and resummation procedure on each remainder a series of higher-order hyperasymptotic corrections is then generated [16,17]. It is clear that, since at each hyperasymptotic step each asymptotic series is shorter than its predecessor, and eventually contains only one term, H necessarily comes at a natural halt [16,17,22]. As a consequence the accuracy level that can be reached with H is lower bounded. For instance, in the particular case of only two involved saddle, as for the Airy function, such a lower bound turns out of the order of $\exp[-(1+2\ln 2)|kF_{+-}|]$ [16]. On comparing the results of Tables I and III with those reported in Table 1 of Ref. [16] and Table 1 of Ref. [17], respectively, it is seen how WT provides relative errors which are orders of magnitude smaller than the corresponding hyperasymptotics bounds. This, of course, is true only for the nonpathological cases examined so far. The situation is completely upset in the pathological cases above described where, differently from the WT, H appears to be rather insensitive with respect to the "input data," as can be realized by comparing the results in Tables 1 and 2 of Ref. [16], corresponding to the same experiments performed in Tables I and II of the present paper, respectively.

For what we seen and said so far, it appears that the WT and H possess features that make them, in a sense, complementary. However, rather than using them separately, we wonder whether their use could be arranged in a joint fashion, in order to conceive a resummation method which could be endowed with the main virtues of both approaches in terms of convergence speed, implementation ease, and insensitivity with respect to the input data. We believe the answer to be in the affirmative. In particular, the idea we are going to pursue in the rest of the present paper is to use, when dealing with the WT pathologies, *only the first stage* of H [see Eq. (24)] as a preliminary step before the WT to be applied. Accordingly, only the saddles adjacent to z_n will be involved, and the single terms of all the asymptotic series in Eq. (24) will be obtainable via simple closed-form expressions. This, in particular, will keep the computational effort and the implementation ease at the same level of that pertinent to the WT which, of course, represents one of the main tasks of the present work. Furthermore, the hope is that, as far as the pathological cases we have examined in Sec. III are concerned, each asymptotic series associated to the remainder could be efficiently resummed via the WT. It should also be stressed that, differently from H, no optimal truncation on the starting series is performed, but rather the order N is left as a free parameter. This represents an important point, because, in order for the WT to be applied to the asymptotic series in Eq. (24), the values of the index r are upper bounded by N. In the next section, for the sake of clarity, the cases of the Airy and the Pearcey functions will be treated separately.

TABLE V. Joint action of H-WT for the pathological case treated in Table II for the numerical evaluation of the Airy function at $x=(3/4\times16)^{2/3}\exp(i2\pi/3^-)$, i.e., across the Stokes line. First column: truncation order. Second column: relative error obtained via H-WT. The number of digits used in the numerical calculations was 22. The same numerical experiment was done in Ref. [16] (see Table 2 therein).

N	Rel. error
2	5×10^{-5}
3	9×10^{-7}
4	2×10^{-8}
5	5×10^{-10}
6	10^{-11}
7	5×10^{-13}
8	2×10^{-14}
9	8×10^{-16}
10	3×10^{-17}
11	2×10^{-17}
12	6×10^{-18}
13	2×10^{-18}
14	8×10^{-19}
15	3×10^{-19}
16	10^{-19}
17	6×10^{-20}
18	3×10^{-20}
19	2×10^{-20}
20	0

VI. NUMERICAL RESULTS

A. Airy function

Consider again the second experiment done in Sec. III B, i.e., the evaluation of the Airy function at the Stokes line. As seen in Sec. III B, of the two contributions $\mathcal{I}^{(\pm)}(1)$, only that pertinent to $z_+\simeq-1.9827\ldots+i1.14471\ldots$ has to be recalculated via the H-WT because of the nonalternating character of the sequence $\{T_r^{(+)}\}$. In particular, $\mathcal{A}_+=\{-\}$, where $z_-=-z_+$, while it turns out that $\gamma_{+-}=0$. In Table V the behav-

TABLE VI. The same as in Table V, but for the numerical evaluation of the Pearcey function at the pair $(x,y)=(7,1/\sqrt{2})$. First column: truncation order. Second column: relative error, obtained via H-WT, evaluated with respect to the 20-digits "exact" value provided in Ref. [26].

N	Rel. error
2	$\times 10^{-4}$
3	3×10^{-6}
4	7×10^{-8}
5	2×10^{-9}
6	6×10^{-11}
7	2×10^{-12}
8	9×10^{-14}
9	2×10^{-15}
10	3×10^{-16}
11	8×10^{-17}
12	10^{-17}
13	3×10^{-18}
14	3×10^{-19}
15	10^{-19}
16	10^{-19}
17	0

ior of the relative error, obtained for different values of the truncation order N, is reported.

By comparing these results to Table II, it is now clear how the H-WT turns out to be quite efficient in evaluating the Airy function with good accuracies. A further comparison could also be done with the results presented in Table 2 of Ref. [16], where the same numerical experiment was carried out via a full H. It is seen at once, as hopefully expected, that the H-WT seems to have gained the virtues of both approaches, namely the insensitivity with respect to the initial data (H) and a good computational efficiency (WT).

B. Pearcey function

As far as the evaluation of the Pearcey function is concerned, we present the results obtained on three experiments carried out on the pairs $(x,y)=(7,1/\sqrt{2})$, $(x,y)=(-4,12/\sqrt{2})$, and $(x,y)=(-7,1/\sqrt{2})$. In particular, the results obtained on the first pair by directly using the WT were previously reported in Table IV of Sec. III C. Moreover, all numerical experiments have also been considered in Ref. [26], so that, as done for the Airy function, we shall use the results reported there for comparison purposes. We start with the first pair, for which an interpretation of the failure of the WT has been given in Sec. IV. We recall that a single saddle does contribute to the integral, and that the inability of the WT to properly work is due to the presence of a *pair* of adjacent saddles endowed with the same modulus of the corresponding singulants. The results of the H-WT are shown, for the present case, in Table VI, where the relative error is reported as a function of the truncation order N. Also in the present case, similarly to what happened for the Airy function evaluation, the H-WT approach allows relative errors of the order of 10^{-20} to be reached with a relatively small truncation orders ($N<20$). The explanation for such a successful operation can still be led back to Eq. (24), from which it is seen that in the two asymptotic series involved in the first order remainder $R^{(2)}(k,N)$ the expanding coefficients corresponding to the saddles z_1 and z_3 display an alternating behavior since, *for each of them*, only one dominant adjacent saddle is present.

TABLE VII. The same as in Table V, but for the numerical evaluation of the Pearcey function at the pair $(x,y)=(-4,12/\sqrt{2})$. First column: truncation order. Second column: relative error, obtained via H-WT, evaluated with respect to the 20-digits "exact" value provided in Ref. [26]. The truncation order N refers only to the H-WT joint action for the evaluation of the contribution corresponding to z_1, while that corresponding to the saddle z_2 has been directly evaluated via a 35-order WT.

N	Rel. error
2	8×10^{-5}
3	2×10^{-6}
4	2×10^{-8}
5	4×10^{-10}
6	2×10^{-11}
7	3×10^{-12}
8	2×10^{-13}
9	4×10^{-14}
10	2×10^{-15}
11	5×10^{-16}
12	4×10^{-17}
13	10^{-17}
14	10^{-18}
15	3×10^{-19}
16	4×10^{-20}
17	0

The second numerical experiment concerns with the evaluation of the Pearcey function at the pair $(x,y)=(-4,12/\sqrt{2})$ (see Table VII). In this case there are *two* contributive saddles, and precisely $z_1\simeq -2.67752\ldots$ and $z_2\simeq 1.33876\ldots+i1.17338\ldots$, while the third (noncontributive) saddle is $z_3=z_2^*$. Accordingly, $P(x,y)=\mathcal{I}^{(1)}(1)+\mathcal{I}^{(2)}(1)$. Moreover, the saddle topology is such that the contribution $\mathcal{I}^{(2)}$ can be retrieved via a direct use of the WT, due to the presence of just one dominant saddle, and precisely z_3. It has been found (not showed for brevity) that a 35-order WT is sufficient to obtain the required 20-digits accurate estimate for it. As far as the contribution $\mathcal{I}^{(1)}$ is concerned, the direct application of the WT is expected to fail, due to the presence of the adjacent pair of complex conjugate saddles (z_2 and z_3), which exactly reproduces the same pathological situation studied in the previous numerical experiment. By using the H-WT approach, however, it is found that, similarly as in Table VI, a truncation order $N=17$ is sufficient to recover, together with the above estimate of $\mathcal{I}^{(1)}$, the 20-digits value provided in Ref. [26]. For the sake of completeness, we also give the results obtained for the third pair considered in Ref. [26], i.e., $(x=-7, y=1/\sqrt{2})$. In the present case all three saddles, $z_1 \simeq -2.694\,88\ldots$, $z_2 \simeq 0.101\,163\ldots$, and $z_3 \simeq 2.593\,72\ldots$ becomes contributive, so that $P(x,y)=\mathcal{I}^{(1)}(1)+\mathcal{I}^{(2)}(1)+\mathcal{I}^{(3)}(1)$. However, as far as z_1 and z_3 are concerned, they have just one adjacent saddle, being $\mathcal{A}_1=\mathcal{A}_3=\{2\}$, while in the case of z_2, for which $\mathcal{A}_2=\{1,3\}$, the saddle z_3 turns out to be dominant with respect to z_1. Since no involved singulant turns out to be real and positive (no Stokes phenomenon occurs), each of the above contributions can be directly computed through the WT. In particular, we have found (not shown in the paper) that a 25-order WT applied on each asymptotic series is sufficient to recover the 20-digits value reported in Ref. [26].

Before concluding this section, however, we also want to give an example of asymptotic evaluation of the Pearcey function at a point (x,y) belonging to the Stokes set defined in Sec. III C. In particular, we choose $x=5$ and $y=\sqrt{\frac{5+3\sqrt{3}}{27}5^3}$. The topology consists in two contributive saddles, $z_1 \simeq -1.104\,57\ldots$ and $z_2 \simeq 0.552\,287\ldots + i2.432\,09$, while the third one is $z_3 = z_2^*$. Accordingly, $P(x,y)=\mathcal{I}^{(1)}(1)+\mathcal{I}^{(2)}(1)/2$, where the factor $1/2$ is due to the presence of the Stokes phenomenon at z_2 which, in turn, leads to a singulant matrix $\{F_{nm}\}$ purely real. The evaluation of the contribution $\mathcal{I}^{(2)}(1)$ does not present any problem, since $\mathcal{A}_2=\{1,3\}$ and z_1 dominates, with $F_{21}<0$. In particular, it turns out that a 17-order WT is enough to recover $\mathcal{I}^{(2)}(1)$ up to 20 digits. As far as the evaluation of $\mathcal{I}^{(1)}(1)$ is concerned, $\mathcal{A}_1=\{2,3\}$, with $F_{12}=-F_{13}>0$, and $\gamma_{12}=0$ and $\gamma_{13}=1$. The results of the application of the H-WT are shown in Table VIII.

TABLE VIII. The same as in Table VII, but for the numerical evaluation of the Pearcey function at the pair $(x,y)=(5,\sqrt{\frac{5+3\sqrt{3}}{27}5^3})$. The relative error obtained via H-WT is evaluated with respect to the 20-digits "exact" value provided by summing the convergent series defining the Pearcey function as explained in Ref. [13]. The truncation order N refers only to the H-WT joint action for the evaluation of the contribution corresponding to z_2, while that corresponding to the saddle z_1 has been directly evaluated via a 17-order WT.

N	Rel. error
2	9×10^{-5}
3	3×10^{-7}
4	3×10^{-8}
5	2×10^{-10}
6	2×10^{-11}
7	3×10^{-13}
8	3×10^{-14}
9	6×10^{-16}
10	4×10^{-17}
11	5×10^{-18}
12	3×10^{-18}
13	2×10^{-18}
14	7×10^{-19}
15	9×10^{-20}
16	5×10^{-20}
17	5×10^{-20}
18	2×10^{-20}
19	10^{-20}
20	0

C. A couple of examples from quantum physics

As a further numerical experiment, we study the asymptotic evaluation of the integral

$$N(k) = k^{1/2}\int_{-\infty}^{+\infty} \exp[-k(z^2-1)^2]dz, \tag{27}$$

for $k>0$, already considered in Ref. [29] as a simplified protoype for the modeling of instanton tunneling between symmetric double wells. As far as the scope of the present paper is concerned, the integral in Eq. (1) is interesting on its own, due to the fact that it displays a Stokes phenomenon, and will be used as a further numerical test for the H-WT. With reference to Eq. (1), the integral in Eq. (27) can be written in the following form:

$$N(k) = 2k^{1/2}\,\mathrm{Re}\{\mathcal{I}(k)\}, \tag{28}$$

with $g(z)=1$, $f(z)=(z^2-1)^2$, and where $\mathcal{C}$ is a steepest descent path, connecting the points $-i\infty$ and $+\infty$, obtained by joining the line $\mathrm{Im}\{z\}\leqslant 0$ and the line $\mathrm{Re}\{z\}\geqslant 0$. There are three saddles, $z_1=-1$, $z_2=0$, and $z_3=1$, but only two of them, z_2 and z_3, do contribute to $\mathcal{I}(k)$. As we shall see in a moment $\mathcal{I}^{(2)}(k)$ turns out to be purely imaginary, so that the values of $N(k)$ are determined only by $\mathcal{I}^{(3)}(k)$, but the corresponding asymptotic series turns out to be nonalternating, being $T_r^{(3)}=2^{-2r-1}(2r-1/2)!/r!$. Thanks to the resurgence-based interpretative scheme, by taking Eq. (21) and the fact that $F_{32}=1$ into account, it is clearly seen that, although not directly involved in the asymptotic evaluation of $N(k)$, the adjacency of z_2 to z_3 is responsible for the nonalternating divergent character of the asymptotic series defining $\mathcal{I}^{(3)}(k)$. The implementation of the H-WT requires the knowledge of the expanding coefficients $T_r^{(2)}$, whose expression follows from Eq. (9) with $U_2(z)=z^2-2$, and after long but straightforward algebra turns out to be

$$T_r^{(2)} = (-1)^{r+1} i \sqrt{\frac{\pi}{2}}\frac{(r-1/2)!}{(2r)!(-2r-1/2)!}, \tag{29}$$

where use has been made of the formula

$$\frac{1}{(2r)!}\left[\frac{d^{2r}}{du^{2r}}\frac{1}{(u^2-u_0^2)^{r+1/2}}\right]_{u=0} = (-1)^{r+1} i \sqrt{\frac{\pi}{2}}\frac{1}{(u_0/\sqrt{2})^{4r+1}(2r)!(-2r-1/2)!}. \tag{30}$$

The results obtained by applying the H-WT for retrieving $N(k)$ are illustrated in Fig. 2, which has been conceived for a direct comparison with Fig. 4 of Ref. [29] to be easily achieved. In particular, in Fig. 2(a) the behavior of $N(k)$ obtained with the use of the H-WT (dots) is plotted, as a func-

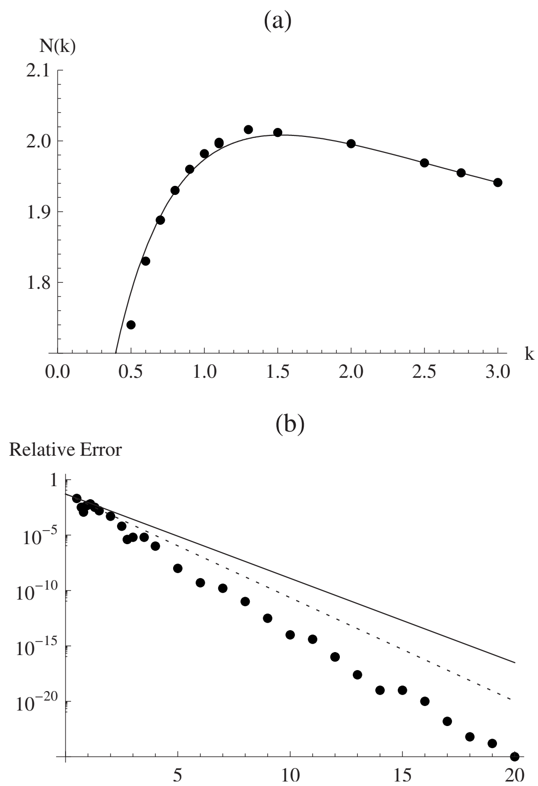

FIG. 2. Numerical evaluation of the function $N(k)$ via the H-WT. (a) Behavior of $N(k)$ obtained, for $k \in [1/2,3]$, with the use of the H-WT (dots) and through the exact expression in Eq. (31) (solid curve). (b) Behavior, as a function of k, of the relative error, evaluated with respect to the exact value of Eq. (31), obtained via the use of the H-WT (dots). For comparison purposes, the behaviors of the relative error corresponding to first and second hyperasymptotic iteration are also reported, as a solid and dotted line, respectively, as extrapolated from Fig. 4b of Ref. [29].

tion of k, together with that (solid curve) corresponding to its exact expression, which turns out to be

$$N(k) = \frac{\pi\sqrt{k}}{2} \exp(-k/2)\left[I_{-1/4}\left(\frac{k}{2}\right) + I_{1/4}\left(\frac{k}{2}\right)\right], \qquad (31)$$

where $I_n(\cdot)$ denotes the nth-order modified Bessel function of the first kind. Note that the range of values of k has been chosen in order to study the performances of the H-WT very far from the asymptotic regime. Figure 2(b) shows the behavior, as a function of k, of the relative error, evaluated with respect to the exact value of Eq. (31), obtained via the use of the H-WT (dots). For comparison purposes, the behaviors of the relative error corresponding to first and second hyperasympotic iteration are also reported, as a solid and dotted line, respectively, as extrapolated from Fig. 4b of Ref. [29].

It is interesting to put into evidence how the integral in Eq. (27) presents a structure somewhat similar to the following one:

$$E(g) = \int_0^\infty \exp(-x^2 - gx^4)dx, \qquad (32)$$

where $g > 0$, whose evaluation has recently been considered in Ref. [30] as a prototype for zero dimensional ϕ^4 theories.

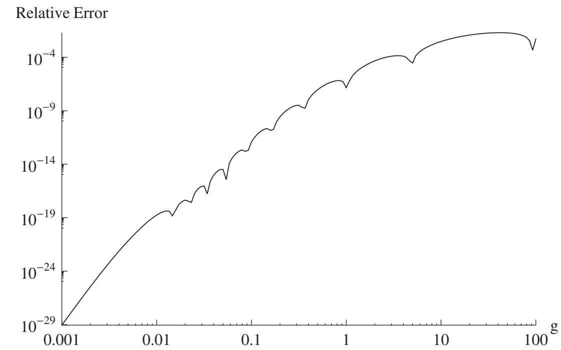

FIG. 3. Behavior of the relative error, evaluated with respect to Eq. (33), for the function $E(g)$, as a function of the parameter g, obtained by using a WT of order 12. This figure should be compared with Fig. 4 of Ref. [30].

Similarly as for the integral in Eq. (27) the function $E(g)$ admits an exact representation, and precisely

$$E(g) = \frac{\exp(1/8g)}{4\sqrt{g}} K_{1/4}\left(\frac{1}{8g}\right), \qquad (33)$$

where $K_n(\cdot)$ denotes the nth-oder modified Bessel function of the second kind. Furthermore, with reference to Eq. (1), the function $E(g)$ can be written as

$$E(g) = \frac{1}{2}\mathcal{I}(1), \qquad (34)$$

with $g(z)=1$, $f(z)=z^2+gz^4$, and where $\mathcal{C}$ is the steepest descent path that coincides with the whole real axis. Also in this case there are three saddles, $z_1=i/\sqrt{2g}$, $z_2=0$, and $z_3=z_1^*$ but, differently from the previous case, only z_2 is contributive. As far as the expanding coefficients $T_r^{(2)}$ are concerned, their behavior for large r can be retrieved directly by applying resurgence in Eq. (20) and by taking into account that $\mathcal{A}_2=\{1,3\}$ and that the singulants $F_{21}=F_{23}=-1/4g$ are real and *negative*. Accordingly, we should expect an alternating factorial divergent behavior. This is confirmed at once by analytically evaluating the coefficients, which, thanks to Eq. (30), turn out to be

$$T_r^{(2)} = (-4g)^r\sqrt{\pi}\frac{(r-1/2)!}{(2r)!(-2r-1/2)!}. \qquad (35)$$

As a matter of fact, we expect that, differently from the previous case, and differently from the pathological case of evaluation of the Pearcey function treated above, characterized by a similar saddle topology, the evaluation of $\mathcal{I}^{(2)}(1)$ could be efficiently carried out by resorting to a low-order WT. How low the order should be is shown in Fig. 3, which could be compared to Fig. 4 of Ref. [30]. In particular, it is seen that in order to achieve relative errors of the same order of magnitude it is sufficient to choose a WT order of 12.

VII. CONCLUSIONS

Since its introduction nearly twenty years ago, the WT has proved to be one of the most efficient and easily implementable tool for the resummation of factorial divergent series. Nevertheless, its systematic use in the steepest descent treatment of saddle-point integrals often is pathological behaviors in which the WT reveals unable to go beyond the superasymptotic estimate provided by the least-term truncation criterion. In the present paper the concept of resurgence has been employed to give a possible explanation of such pathological behaviors when the Airy and the Pearcey functions are asymptotically evaluated. In particular, it has been found that the topology of the *whole* set of the saddle points associated to the function $f(z)$ strongly influences the resummation capabilities of the WT, in the presence of Stokes phenomena as well as when two (or more) adjacent saddles have the same singulant modulus. A powerful, easily implementable, resummation scheme aimed at dealing with the above pathological cases has then been proposed. Such a scheme anticipated, as a preliminary step, to operate a first (not necessarily optimum) truncation on each pathological series and, subsequently, employed the WT only on the asymptotic series which are generated by the first-stage hyperasymptotic treatment of the corresponding diverging remainder. This joint action (the H-WT) seems, at least for the cases numerically investigated, to keep the main virtues of both approached, namely a relatively insensitivity with respect to the "initial data" (characteristic of H) and an implementation ease and a convergence speed typical of the WT. The preliminary results obtained here should encourage a deeper investigation, both theoretical and numerical, toward the study of the mechanisms of the H-WT and its application to the asymptotic evaluation of integrals characterized by more complex saddle points topologies such as, for instance, those involved in the study of higher-order diffraction catastrophes (swallowtail function, elliptical and hyperbolic umbilics) [31] or those recently considered in Refs. [32,33] as far as the scattering of whispering-gallery modes is concerned. The study of the interaction between hyperasymptotic stages of order higher than 1 and the WT would also be of interest.

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